Quantum Field Theory in Terms of Non-Standard Mathematics (ハミルトニアンの定義とスペクトル)

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Abstract

The mathematical structure of the quantum field theory is investigated with the help of the so-called non-standard mathematics. It is seen that there exists a renormalized Hamiltonian defined as a self-adjoint operator, and that it determines the S-matrix almost uniquely. The perturbation expansion of the S-matrix converges absolutely at least in the non-standard sense.

1. Introduction

As it is well known the quantum field theory, from the beginning of its birth, has fatalistic difficulties called the difficulties of divergence. Those difficulties are partially removed by the renormalization theory. Especially, in quantum electrodynamics the agreement between experimental and theoretical values is almost surprising. Such agreement allows us to presuppose that the renormalization theory provides a true explanation of the nature, at least in quantum electrodynamics.

The main purpose of this article is to make a first step to the proof of this presumption based on a mathematically rigorous foundation with respect to the formal and/or willful operations arising in the renormalization theory.

We shall begin, at first, to construct a field theoretical model described in terms of non-standard mathematics[1]. This model enables us to give a rigorous meaning to the divergences in the perturbation expansion. The renormalized Hamiltonian of such a model, defined as a self-adjoint operator, determines the S-matrix almost uniquely, and the perturbation series converges absolutely, at least in the sense of non-standard mathematics. However, we are left with the problem whether the resultant series converges rapidly so that the first few terms may give a good approximation. The absolute convergence of the series is proved in terms of the non-standard language, and hence the proof of it in the standard sence is still an open question.

In the following discussions we make use of φ^4 scalar theory for simplicity. Essentially same results can be obtained in other renormalizable theories, especially in quantum electrodynamics.

2. Non-standard Hamilton models

Let ω be a non-standard natural number[1] greater than any other standard numbers, and let $\varphi(x)$ and $\pi(x)$ be the canonical field variables such that

$$\varphi(\vec{x}) = \sum_{\alpha=1}^{\omega} q_\alpha e_\alpha(\vec{x}), \quad \pi(\vec{x}) = \sum_{\alpha=1}^{\omega} p_\alpha e_\alpha(\vec{x})$$

$$[p_\alpha, q_\beta] = -i\delta_{\alpha\beta}, \quad [p_\alpha, p_\beta] = 0, \quad [q_\alpha, q_\beta] = 0 \tag{2. 1}$$

where $\{e_\alpha(\vec{x})\}$ is a complete set of orthonormal functions over $({}^*R)^3$. The canonical variables $\varphi(\vec{x})$ and $\pi(\vec{x})$ satisfy the canonical commutation relations as operator-valued $(\hat{\mathcal{S}}')_{R^3}$ *) functions:

$$[\pi(\vec{x}), \varphi(\vec{y})] \doteq -i\delta(\vec{x}-\vec{y})$$

$$[\pi(\vec{x}), \pi(\vec{y})] = 0, \quad [\varphi(\vec{x}), \varphi(\vec{y})] = 0. \tag{2. 2}$$

The total Hamiltonian H is then defined as

$$H = \frac{1}{2}\int d^3x \{(\pi(\vec{x}))^2 + (\vec{\nabla}\varphi(\vec{x}))^2 + m^2(\varphi(\vec{x}))^2\} + H_I \tag{2. 3}$$

where H_I is a certain function of φ and is bounded below. In §4 we shall deal with the specified renormalized interaction Hamiltonian

$$H_I = g\int d^3x (\varphi(\vec{x}))^4 + \delta m^2 \int d^3x (\varphi(\vec{x}))^2 + \delta g \int d^3x (\varphi(\vec{x}))^4$$

which really satisfies the condition stated above. Because of the orthonormality of $\{e_\alpha(\vec{x})\}$, we can rewrite the terms in (2. 3) as

*) See Def. 4 of Math. Appendix.

$$\int d^3x \, (\pi(\vec{x}))^2 = \sum_{\alpha=1}^{\omega} p_\alpha^{\ 2}$$

$$\int d^3x \, (\vec{\nabla}\varphi(\vec{x}))^2 = \sum_{\alpha,\beta=1}^{\omega} C_{\alpha\beta}\, q_\alpha q_\beta \tag{2. 4}$$

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Let $*L^2(G)$ be the non-standard Hilbert space of the square-integrable functions on G, and make a realization of $\{p_\alpha, q_\alpha\}$ in it by $p_\alpha = -i\frac{\partial}{\partial \xi_\alpha}$ and $q_\alpha = \xi_\alpha$. The Hamiltonian H then becomes

$$H = -\frac{1}{2}\Delta + U(\xi) \tag{2. 5}$$

where Δ is a Laplacian in ω-dimensional Euclidian space $({}^*\mathbb{R})^\omega$. If we restrict G to a certain (standard) compact region in $({}^*\mathbb{R})^\omega$, then $U(\xi)$ is a continuous function of $\xi = (\xi_1 \cdots \xi_\omega)$ and is bounded below.

The proof of the existence of nontrivial examples of Hamilton model comes from the fact that there exist a non-trivial solution of the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(\xi t) = \left(-\frac{1}{2}\Delta + U(\xi)\right)\Psi(\xi t). \tag{2. 6}$$

In fact, under certain conditions, the boundedness of U enables us to show that the solution of this equation does exist uniquely. For example, we can treat it as a mixed problem. Let us put the initial and boundary condition as

$$\Psi(\xi 0) = f(\xi)\,,\quad \xi \in G\,,\quad f(\xi) \in {}^*L^2(G)$$
$$\left(\alpha \Psi + \beta \frac{\partial}{\partial n} \Psi\right)\Big|_{\partial G} = 0 \qquad (2.\ 7)$$

Then the solution of (2. 6) and (2. 7) can be written as

$$\Psi(\xi t) = \sum_{k=1}^{\infty} a_k\, e^{-i\lambda_k t}\, \psi_k(\xi) \qquad (2.\ 8)$$

where λ_k's and ψ_k's are the eigenvalues and eigenfauncions, respectively, of the equation

$$\left(-\frac{1}{2}\Delta + U(\xi) - \lambda_k\right)\psi_k = 0 \qquad (2.\ 9)$$

Since the ψ_k's satisfy the boundary condition (2. 7) and since the totality of the eigensolutions of (2. 9) is known on $^*L^2(G)$[2), the expansion (2. 8) with $a_k = (f, \psi_k)$ is the unique solution of (2. 6) and (2. 7).

The Hamiltonian H diagonalized in $^*L^2(G)$ is a self-adjoint operator. Therefore the operator $U_t \equiv \exp(-iHt)$ is unitary. It connects the Scrödinger picture to the Heisenberg picture:

$$\begin{aligned} \Psi(\xi t) &\longrightarrow U_t^{-1}\Psi(\xi t) \equiv \Psi(\xi) \\ \varphi(\vec{x}) &\longrightarrow U_t^{-1}\varphi(\vec{x})\,U_t \equiv \varphi(\vec{x}, t) \qquad (2.\ 10) \\ \pi(\vec{x}) &\longrightarrow U_t^{-1}\pi(\vec{x})\,U_t \equiv \pi(\vec{x}, t) \end{aligned}$$

in which the Heisenberg variables π(x, t) and φ(x, t) satisfy the equal-time commutation relation as operator-valued $(\hat{\mathcal{S}}')_{R^3}$ functions

$$[\,\pi(\vec{x},t),\ \varphi(\vec{y},t)\,] \doteq -i\,\delta(\vec{x}-\vec{y})$$

$$[\,\pi(\vec{x},t),\ \pi(\vec{y},t)\,] = 0\ ,\quad [\,\varphi(\vec{x}\,t),\ \varphi(\vec{y}\,t)\,] = 0. \tag{2. 11}$$

It is worth noticing that there also exists an interaction picture in our model, which we shall powerfully utilize in the discussions in §§3 and 4. Let H_0 be a free Hamiltonian

$$\begin{aligned} H_0 &= \frac{1}{2}\int d^3x\,\{\,(\pi(\vec{x}))^2 + (\vec{\nabla}\varphi(\vec{x}))^2 + m^2(\varphi(\vec{x}))^2\,\} \\ &= -\frac{1}{2}\Delta + V(\xi) \end{aligned} \tag{2. 12}$$

It is easily seen that the second term is bounded below (positive definite) and the solution of the eigenvalue equation

$$(\,H_0 - \varepsilon_k\,)\,\eta_k = 0 \tag{2. 13}$$

is dense in $*L^2(G)$. Hence we can expand the solution of (2. 6) ans (2. 7) by $\{\eta_k\}$ as

$$\Psi(\xi\,t) = \sum_{k=1}^{\infty} c_k(t)\,\eta_k(\xi) \tag{2. 14}$$

Let us observe the time dependence of $b_k(t) \equiv c_k(t)e^{i\varepsilon_k t}$:

$$i\,\dot{b}_k(t) = \sum_{\ell=1}^{\infty} b_\ell(t)\,e^{i(\varepsilon_k-\varepsilon_\ell)t}\,(\eta_k,\ H_I\,\eta_\ell) \tag{2. 15}$$

Now, if we define a new operator $H_I(t) \equiv e^{iH_0t}H_I\,e^{-iH_0t}$, we finally arrive at the wave equation in the interaction picture:

$$i\frac{\partial}{\partial t}\Psi(t) = H_I(t)\,\Psi(t) \tag{2. 16}$$

$$\Psi(t) = \sum_{k=1}^{\infty} b_k(t)\,\eta_k\,.$$

The operator e^{iH_0t} is unitary since H_0 is self-adjoint.

For a given standard theory, different choices of ω , $\{e_\alpha(\vec{x})\}$ or $*L^2(G)$ lead to various models which are, however, physically equivalent with each other, and which give almost same S-matrix if renormalization is accomplished.

3. Realization of standard theory

Let $\varphi(x)$ and $\pi(x)$ be the canonical field variables (in a certain model) defined in the preceding section, and let us take off the so-called zero point energy W_0 in order that the lowest eigenvalue of H_0 is exactly zero.

$$H_0 = \frac{1}{2}\int d^3x \left\{(\pi(\vec{x}))^2 + (\vec{\nabla}\varphi(\vec{x}))^2 + m^2(\varphi(\vec{x}))^2\right\} - W_0 \qquad (3.\ 1)$$

The time dependence of the operators in the interaction picture is the following:

$$\varphi(x) = e^{iH_0t}\varphi(\vec{x})\,e^{-iH_0t}, \quad \pi(x) = \dot{\varphi}(x). \qquad (3.\ 2)$$

The expression of $\ddot{\varphi}(x)$ is not so simple because of the $\vec{\nabla}$'s in the integrand of H_0. However, we can show the following theorem:

[Theorem 1] The operator $\ddot{\varphi}(x)$ is almost equal to $(\nabla^2 - m^2)\varphi(x)$ as an operator-valued $(\hat{\mathcal{S}}')_{R^4}$ function, with a suitable choice of the bases $\{e_\alpha(\vec{x})\}$:

Proof: Since $(\hat{\mathcal{S}})_R \times (\hat{\mathcal{S}})_{R^3}$ is total in $(\hat{\mathcal{S}})_{R^4}$, we may restrict ourselves to $f(x) = u(t)v(\vec{x})$.

$$\ddot{\varphi}(f) = i\,[\,H_0,\ \pi(f)\,]$$

$$= -\int dx \int_{x_0'=x_0} d^3x' \left\{ (\vec{\nabla}'\varphi(x')) \cdot \vec{\nabla}' \sum_{\alpha=1}^{\omega} e_\alpha(\vec{x}') e_\alpha(\vec{x}) f(x) \right.$$

$$\left. + m^2 \varphi(x') \sum_{\alpha=1}^{\omega} e_\alpha(\vec{x}') e_\alpha(\vec{x}) f(x) \right\} \tag{3. 3}$$

Now $\varphi(x) = \sum_{\alpha=1}^{\omega} g_\alpha(t)\ e_\alpha(\vec{x})$ and $f(x) = u(t) \sum_{\alpha=1}^{\infty} v_\alpha\ e_\alpha(\vec{x})$ where v_α's are rapidly decreasing with respect to α in the standard sense. Hence we have

$$\int dx \int_{x_0'=x_0} d^3x'\ \varphi(x') \sum_{\alpha=1}^{\omega} e_\alpha(\vec{x}') e_\alpha(\vec{x}) f(x) = \int dx\ \varphi(x) f(x)$$

$$\int dx \int_{x_0'=x_0} d^3x' (\vec{\nabla}'\varphi(x')) \cdot \vec{\nabla}' \sum_{\alpha=1}^{\omega} e_\alpha(\vec{x}') e_\alpha(\vec{x}) f(x) \tag{3. 4}$$

$$= -\int dx\ \varphi(x) \nabla^2 f(x) + \int dx\ u(t) \sum_{\alpha=\omega+1}^{\infty} v_\alpha\ e_\alpha(\vec{x}) \nabla^2 \varphi(x)$$

Let us try to estimate the last term:

$$\left\| \int dx\ u(t) \sum_{\alpha=\omega+1}^{\infty} v_\alpha\ e_\alpha(\vec{x})\ \nabla^2 \varphi(x) \right\|$$

$$= \left\| \sum_{\alpha=\omega+1}^{\infty} \sum_{\beta=1}^{\omega} v_\alpha \int_{-\infty}^{\infty} dt\ g_\beta(t)\, u(t) \int d^3x\ e_\alpha(\vec{x})\, \nabla^2 e_\beta(\vec{x}) \right\| \tag{3. 5}$$

Choosing G of $^*L^2(G)$ satisfying $\max_\beta \| g_\beta \| = Q \in \mathbb{R}$, and observing the equality

$$\| g_\beta(t) \| = \| e^{iH_0 t} g_\beta\, e^{-iH_0 t} \| = \| g_\beta \|$$

we get

$$\text{the r.h.s. of (3. 5)} \leqq Q \cdot \sum_{\alpha=\omega+1}^{\infty} \sum_{\beta=1}^{\omega} |v_\alpha|$$

$$\times \int_{-\infty}^{\infty} dt\ |u(t)| \cdot \left| \int d^3x\ e_\alpha(\vec{x})\, \nabla^2 e_\beta(\vec{x}) \right| \tag{3. 6}$$

Some suitable choice of the basis $\{e_\alpha(\vec{x})\}$ (for example Hermite functions) makes it possible that only a few terms survive in the right hand side of (3. 6) with α's and β's in a finite neighborhood of ω. Since the v_α's are rapidly decreasing, it becomes smaller than any power of $1/\omega$. We conclude, therefore,

$$\varphi((\Box+m^2)f) \doteq 0 \qquad (3.7)$$

on a dense subset of $\mathcal{H}$. ∎

The result obtained above enables us to write $\varphi(x)$ as an operator-valued $(\hat{\mathcal{S}}')_{R^4}$ function such that

$$\varphi(x) \doteq \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int d\Omega_m(p) \left(a(p)\, e^{-ip\cdot x} + a^\dagger(p)\, e^{ip\cdot x} \right) \qquad (3.8)$$

where $d\Omega_m(p)$ is an invariant measure defined on the hyperboloid $V_m^+ = \{p \mid p^2 = m^2,\ p^0 > 0\}$.

We next prove:

[Theorem 2] The function $\hat{D}(x,x') \equiv [\varphi(x), \varphi(x')]$ is seen to be an "embodiment"*) of the standard Pauli-Jordan function $D(x-x')$.

Proof: It is almost obvious since $\hat{D}(x, x')$ satisfies the following relations

$$(\Box+m^2)\hat{D}(x,x') \doteq 0, \quad \hat{D}(x,x')\big|_{x_0'=x_0} = 0, \qquad (3.9)$$
$$\partial_0 \hat{D}(x,x')\big|_{x_0=x_0'} \doteq -i\,\delta(\vec{x}-\vec{x}'). \ ∎$$

If we define the functions

$$f_\pm(x) = \left(\frac{1}{2\pi}\right)^{\frac{5}{2}} \int dp\, \theta(p^0)\, e^{\pm i p\cdot x}\, \tilde{f}(p), \qquad (3.10)$$
$$\tilde{f}(p) \in (\hat{\mathcal{S}})_{R^4},$$

it is easy to observe that

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*) See Definition 2 of the Mathematical Appendix.

$$\varphi(f_-) \doteq \int d\Omega_m(p)\, \tilde{f}(p)\, a^+(p),$$
$$\varphi(f_+) \doteq \int d\Omega_m(p)\, \tilde{f}(p)\, a(p)\ . \tag{3. 11}$$

We have on the one hand

$$[\varphi(f_+),\ \varphi(g_-)] \doteq \int d\Omega_m(p) \int d\Omega_m(q)$$
$$\times\ \tilde{f}(p)\, \tilde{g}(q)\, [\, a(p)\, ,\ a^+(q)\,] \tag{3. 12}$$

and on the other hand the relation $\hat{D}(x, x') \doteq D(x-x')$ assures

$$[\varphi(f_+),\ \varphi(g_-)] \doteq \int dx \int dx'\ f_+(x)\, D(x-x')\, g_-(x') \tag{3. 13}$$
$$\doteq \int d\Omega_m(p)\ \tilde{f}(p)\, \tilde{g}(p)\ .$$

Hence we conclude

$$[\, a(p)\, ,\ a^+(q)\,] \doteq 2\omega_p\ \delta(\vec{p}-\vec{q})\ . \tag{3. 14}$$

Similarly we get

$$[\, a(p)\, ,\ a(q)\,] \doteq 0\ , \quad [\, a^+(p)\, ,\ a^+(q)\,] \doteq 0\ ,$$
$$[\, a^+(p)\, ,\ a(q)\,] \doteq -2\omega_p\ \delta(\vec{p}-\vec{q})\ . \tag{3. 15}$$

With the help of the definition of $\dot{\varphi}(x)$ and the expression of

$\varphi(x)$ the following relations hold as relations between operator-valued $(\hat{\mathcal{S}}')_{V_m^+}$ functions

$$[H_0, a^\dagger(p)] \doteqdot \omega_p a^\dagger(p)$$
$$[H_0, a(p)] \doteqdot -\omega_p a(p) \tag{3. 16}$$

Hence we can interpret the $a^\dagger(p)$'s and $a(p)$'s as creation and annihilation operators, as is usually done in the standard theory.

If we write Φ_0 for the (non-degenerate) eigenvector belonging to the lowest eigenvalue $E_0 = 0$ of H_0, we may, with the help of (3. 16), interpret the states $\varphi(f_-)\Phi_0$ as one particle states. In general the vectors $\varphi(f_-^{(1)}) \cdots \varphi(f_-^{(n)})\Phi_0$ represent the embodiments of n-particle states. On the other hand, because of the positive-definiteness of H_0 the vectors $\varphi(f_+^{(1)}) \cdots \varphi(f_+^{(n)})\Phi_0$ must be null. It can be easily seen that the space $\hat{\mathcal{H}}$ 3), which is generated in the standard way from Φ_0 by the $\varphi(f_-)$'s, is the embodiment of the standard Fock space $\mathcal{H}_F$.

We give here, without proof, the relations which hold for $(\hat{\mathcal{S}}')_{R^4} \times (\hat{\mathcal{S}}')_{R^4} \times \cdots$ functions:

$$(\Phi_0, T\,\varphi(x)\varphi(x')\Phi_0) \doteqdot D_F(x-x')$$
$$(\Phi_0, T\varphi(x_1)\cdots\varphi(x_n)\Phi_0) \tag{3. 17}$$
$$\doteqdot \frac{1}{n}\sum_{j<k} D_F(x_j - x_k)\,(\Phi_0, T\varphi(x_1)\cdots \overset{j}{\vee} \cdots \overset{k}{\vee} \cdots \varphi(x_n)\Phi_0)$$

where the symbol $\overset{j}{\vee}$ indicates the omission of the factor $\varphi(x_j)$.

In such a way one can construct all the relations which appear in the standard theory described in the interaction picture. If we take, at this stage, the quotient of our model by the relation

$\stackrel{\bullet}{=}_\bullet$, we would go over into the standard φ^4 theory which is mathematically vague. At the same time there reappear the difficulties usually observed. The reason why those difficulties reappear is now obvious: the interaction does not keep $\hat{\mathcal{H}}$ invariant and we must carry it on in the total space $\mathcal{H}$, but on the other hand the relation $\stackrel{\bullet}{=}$, is not a congruence relation in $\mathcal{H}$, so that we can not catch it up with the standard mathematics. That is the reason why we must consider the non-standard models. We believe that without the help of these models we can not clear up the mathematical structure of the quantum field theory and grasp the meaning of the renormalization procedure.

4. Renormalization

Let us begin to solve the wave equation

$$i\frac{\partial}{\partial t}\Psi(t) = H_I(t)\Psi(t) \tag{4. 1}$$

which is established in §2.

The wave operator $U(t, t_o)$ defined by $\Psi(t) = U(t, t_o)\Psi(t_o)$ satisfies the equation

$$i\frac{\partial}{\partial t}U(t,t_o) = H_I(t)\,U(t,t_o) \tag{4. 2}$$

The formal solution of Eq. (4. 2) is given by

$$U(t,t_o) = 1 + \sum_{n=1}^{\infty}(-i)^n\int_{t_o}^{t}dt_1\cdots\int_{t_o}^{t_{n-1}}dt_n\,H_I(t_1)\cdots H_I(t_n) \tag{4. 3}$$

where we have put the initial condition $U(t_o, t_o) = 1$.

[Theorem 3] The formal solution (4. 3) is really the solution of Eq. (4. 2).

Proof: Since H_I is a polynomial of $\xi = (\xi_1 \cdots \xi_\omega)$ and since

$$\| H_I(t) \| = \| e^{iH_0 t} H_I e^{-iH_0 t} \| = \| H_I \|$$

the compactness of G of $*L^2(G)$ guarantees the existence of $D \in {}^*R$ such that $\| H_I \| \leqq D$. Hence there holds the estimation

$$\left\| \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n \, H_I(t_1) \cdots H_I(t_n) \right\| \leqq \frac{1}{n!} (t-t_0)^n D^n$$

The right hand side of this inequality vanishes as $n \to \infty$. ■

In the standard φ^4 theory[4)] each term of the perturbation expansion of S-matrix is finite if we adopt the so-called renormalized interaction Hamiltonian

$$H_I(t) = g \int d^3x \, (\varphi(x))^4 + \delta m^2 \int d^3x \, (\varphi(x))^2 + \delta g \int d^3x \, (\varphi(x))^4 \qquad (4.\ 4)$$

which is mathematically vague, and if we invoke the completion of the renormalization procedure. Renormalization essentially consists in the rearrangement of the terms under the summation, but since the quantities turning up are always divergent, the procedure has only a formal meaning. We emphasize that the rearrangement is necessary because the last two terms of (4. 4), which serve for the cancellation of the divergences, are regarded as higher order perturbetions in g. for example $\delta m^2 = \sum_{n=1}^{\infty} g^{2n} \mathcal{H}_{2n}$, $\delta g = g \sum_{n=1}^{\infty} g^{2n} \eta_{2n}$ Each term is then uniquely determined according to the physical postulate, the so-called renormalization condition.

The renormalization in the standard theory induces the renormalizations of the embodiments. ~~The~~ Since we already completed the realization of standard theory, the renormalizations of the embodiments are subject to the same prescription as the original ones, but the mathematical structure of those are far clear and simple enough to grasp.

In general, rearrangement of the terms in infinite series critically leads to different results so that they are unreliable to bear physical meanings. We must show, therefore, that the renormalized interaction gives a unique S-matrix, but it is almost impossible to prove this in terms of the standard language.

In the non-standard theory, however, it is easy to show:

[Theorem 4] For a given renormalized Hamiltonian there exists a wave operator $U(T, -T)$ which gives the S-matrix almost uniquely.

Proof: The coefficients $\mathcal{H}_{2n}$ and η_{2n} in

$$\delta m^2 = \sum_{n=1}^{\infty} \mathcal{H}_{2n} g^{2n}, \quad \delta g = g \sum_{n=1}^{\infty} \eta_{2n} g^{2n},$$

are determined by the renormalization condition step by step. Because of the absolute convergence of the unrenormalized perturbation series, those series of δm^2 and δg must be absolutely convergent. It suffices then to consider the terms of the form

$$(-i)^p \int_{-T}^{T} dt_1 \cdots \int_{-T}^{t_{p-1}} dt_p \int d^3x_1 \cdots \int d^3x_p$$
$$\times\, g^l (\delta m^2)^m (\delta g)^n \, (\varphi(x_1))^{\delta_1} \cdots (\varphi(x_p))^{\delta_p},$$

where $p = \ell + m + n$ and q_i takes the values 2 and 4. Since $\delta m^2, \delta g \in {}^*R$ and G of ${}^*L^2(G)$ is compact, we can estimate

$$\| g \int d^3x\, (\varphi(x))^4 \| \leqq |g| \cdot \left| \sum_{\alpha,\beta,\gamma=1}^{\omega} C_{\alpha\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma \right| \equiv A \in {}^*R$$

$$\| \delta m^2 \int d^3x\, (\varphi(x))^2 \| \leqq |\delta m^2| \cdot \left| \sum_{\alpha=1}^{\omega} \xi_\alpha^2 \right| \equiv B \in {}^*R$$

$$\| \delta g \int d^3x\, (\varphi(x))^4 \| \leqq |\delta g| \cdot \left| \sum_{\alpha,\beta,\gamma=1}^{\omega} C_{\alpha\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma \right| \equiv C \in {}^*R$$

We conclude, therefore, that

$$\| (-i)^p \int_{-T}^{T} dt_1 \cdots \int_{-T}^{t_{p-1}} dt_p \int d^3x_1 \cdots \int d^3x_p$$
$$\times\, g^\ell (\delta m^2)^m (\delta g)^n (\varphi(x_1))^{q_1} \cdots (\varphi(x_p))^{q_p} \|$$
$$\leqq \frac{1}{p!} (2T)^p A^\ell B^m C^n$$

The right hand side of this inequality vanishes as $p \to \infty$. ∎

After the renormalization the matrix elements $(\Psi_a, U(T,-T)\Psi_b), (R < T \in {}^*R, \Psi_a, \Psi_b \in \hat{\mathcal{H}})$ present the embodiments of the S-matrix elements. Therefore we can conclude that we have proved the absolute convergence of the S-matrix and, at the same time, its uniqueness,

We make a few remarks at this point.

First we note that we have proved the absolute convergence of the perturbation expansion only in the non-standard model, and it does not necessarily mean the convergence of the original series in the standard sense. It is true that for $\forall n \in \mathbb{N}$ the n-th terms of the original and the embodiment series are in the relation $\dot{=}$, but it is not sure by the argument given above, whther the infinite accumulation of infinitesimal errors remain infinitesimal.

The second remark, which may be easily overlooked, is the following. Most of the relations established in §3 are relations in the sense $\dot{=}$, but this is not the congruence relation in the non-standard calculations. Hence we must, first of all, restrict the integration region of (4. 4) to a certain standard compact region, and also the time interval as $2T \in \mathbb{R}$. If we use the relation $\dot{=}$ instead of the true equality, what we can conclude is the following: let Ω be a four-dimensional square

$$\Omega = \{ x \mid |x^0| \leqq T \in \mathbb{R} ,\ |x_i| \leqq L \in \mathbb{R},\ i = 1, 2, 3 \}$$

and let $\{\Omega_i\}_{i \in \mathbb{N}}$ be a truely increasing sequence of such squares. Then for a given arbitrary positive number $\varepsilon \in \mathbb{R}$ there exists $J \in \mathbb{N}$ such that

$$| U^{(n)}_{ab}(\Omega_i) - U^{(n)}_{ab}(\Omega_j) | < \varepsilon \qquad \text{for} \quad \forall i, \forall j > J$$

where $U^{(n)}_{ab}(\Omega) \equiv (\Psi_a, U^{(n)}_V(T,-T)\, \Psi_b)$ with $\Psi_a, \Psi_b \in \hat{\mathcal{H}}$ $, V = L^3$, is the n-th term of the perturbation expansion of $(\Psi_a, U_V(T,-T)\Psi_b)$. In that case there exists at least one Ω , and therefore innumerably many Ω 's ($\mathbb{R}^4 < \Omega \in (^*\mathbb{R})^4$), for which $U^{(n)}_{ab}(\Omega)$ is the embodiment of $S^{(n)}_{ab}$. It is worthwhile to notice that it is not sure whether the statement above is true or not for arbitrary $\mathbb{R}^4 < \Omega \in (^*\mathbb{R})^4$.

With these remarks in mind we can write down the embodiment of the perturbation expansion of the S-matrix

$$\begin{aligned} U_{ab}(\Omega) &= \delta(a,b) + \sum_{n=1}^{\infty} g^n\, U^{(n)}_{ab}(\Omega) \\ &= \delta(a,b) + \sum_{n=1}^{\infty} g^n \left(S^{(n)}_{ab} + \varepsilon^{(n)}_{ab} \right) \end{aligned} \tag{4. 5}$$

where $S^{(n)}_{ab}$ is the copy*) of the corresponding renormalized term in the standard theory for $n \in \mathbb{N}$, and $\varepsilon^{(n)}_{ab}$ is infinitesimal for $n \in \mathbb{N}$.

5. Discussion

With the help of the non-standard Hamilton model we successfully gave the mathematically rigorous foundation for the renormalization procedure

The series (4. 5) for the S-matrix converges absolutely at least in the non-standard sense, and it is almost uniquely determined by the given Hamiltonian which is rigorously defined as a self-adjoint operator. The S-matrix, on the other hand, does not uniquely determine the Hamiltonian. The collection of those non-standard models which realise the given S-matrix forms an equivalentclass. Any elements of the equivalent class are physically equivalent with each other, that is, those non-standard models reproduce the almost same S-matrix.

It is desirable from the practical point of view that the convergence of the series is rapid enough. The series of the

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*) See Definition 1 in the Mathematical Appendix.

standard φ^3 theory, on the other hand, was pointed out not to be convergent[5]. If it is true, the series corresponding to (4. 5) should not converge in the standard sense in the φ^3 theory. This means that the accumulation of the infinitesimal errors becomes infinitely large, making the definition (4. 5) of the S-matrix meaningless.

There seems to be no definite conclusions up to now on the convergence of the series in other renormalizable theories such as the φ^4 theory or quantum electrodynamics. As for quantum electrodynamics the experiments strongly suggest that the convergence of the series is good. If the theory would predict that it does not converge in the standard sense, we should be confronted with the puzzling problem, why the first few terms agree with experiments so miraculously.

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Mathematical Appendix

[Definition 1] When a object f of standard mathematics is given, we define the copy of f in non-standard mathematics as

$$\{ (f_0, f_1, f_2, \cdots) \mid (f_0, f_1, f_2, \cdots) \equiv (f, f, f, \cdots) \} / \equiv$$

Example: the copy of $1 \in \mathbb{R}$ is $^*1 \in {}^*\mathbb{R}$ which is generated by $(1, 1, \ldots) \in \mathbb{R}^{\mathbb{N}}$. We use a same symbol for the copy and the original.

[Definition 2] The non-standard object, which is in the relation $\doteq$, to the copy of f, is called the embodiment of f. The wording is sometimes used in an extended meaning. Example: (1, 2, 3, ...) is the embodiment of the infinity in standard mathematics.

[Definition 3] The space $(\hat{\mathcal{S}})_{R^n}$, $n \in \mathbb{N}$ is defined as the embodiment of standard Schwartz's space $(\mathcal{S})_{R^n}$. $(\hat{\mathcal{S}})_{R^n}$ is the vector space of complex-valued functions which map $({}^*\mathbb{R})^n$ into $\mathbb{C}$: we can differentiate f(x) any times, and when $|x| = |x_1| + \cdots + |x_n|$ increases greater than any other standard numbers, f(x) and its derivatives decreases faster than any power of $1/|x|$. In other words:

> For any $\alpha = (\alpha_1 \cdots \alpha_n)$ and any $\beta = (\beta_1 \cdots \beta_n), (\alpha_i, \beta_i \in \mathbb{N})$
> we have $| x^\alpha D^\beta f(x) | \doteq 0$ for all x, $\mathbb{R} < |x| \in {}^*\mathbb{R}$.

The topology of $(\hat{\mathcal{S}})_{R^n}$ is defined by the semi-norms

$$p_\sigma(f) = \max_{|\alpha| \leqq \sigma} \max_{|\beta| \leqq \sigma} \sup_{x \in ({}^*\mathbb{R})^n} | x^\alpha D^\beta f(x) | \in \mathbb{R}$$

where $\sigma = 0, 1, 2, \ldots \in \mathbb{N}$. Thus the space $(\hat{\mathcal{S}})_{R^n}$ is invariant under standard operations defined in $(\hat{\mathcal{S}})_{R^n}$, so that the equivalence relation $\stackrel{\bullet}{=}$ becomes a congruence relation in $(\hat{\mathcal{S}})_{R^n}$.

[Definition 4] We define $(\hat{\mathcal{S}}')_{R^n}$ as the embodiment of the tempered distribution space $(\mathcal{S}')_{R^n}$. It is the dual space of $(\hat{\mathcal{S}})_{R^n}$.

Example: the function $(\omega/2\pi)^{n/2} \exp\left(-\frac{\omega}{2}(x_1^2 + \cdots + x_n^2)\right)$ is an element of $(\hat{\mathcal{S}}')_{R^n}$ and is an embodiment of $\delta(x) \in (\mathcal{S}')_{R^n}$.

[Remark] It is necessary not to confuse $(\hat{\mathcal{S}}), (\hat{\mathcal{S}}')$ and $({}^*\mathcal{S}), ({}^*\mathcal{S}')$. The latter are the non-standard extensions of $(\mathcal{S}), (\mathcal{S}')$.

Example: $(\omega/2\pi)^{1/2} \exp(-\frac{\omega}{2}x^2)$ is an element of $({}^*\mathcal{S})_R$ but not of $(\hat{\mathcal{S}})_R$.

References and remarks

1) G. Takeuchi, Proc. Jap. Acad. 38 (1962).
A. Robinson, Non-standard Analysis, North-Holland Publ. Co,, (1965). J. Math. Phys. 13, 1870 ('72).

2) See, for example: V. S. Vladimirov, Partial Differential Equations, Sogo Tosho Publ. Co. (in Japanese).

3) $\hat{\mathcal{H}}$ is a linear aggregate of vectors $\{\Phi_0, \varphi(f)\Phi_0, \varphi(f)\varphi(g)\Phi_0, \cdots\}$. Each element of $\hat{\mathcal{H}}$ has a finite norm in the standard sense. $\hat{\mathcal{H}}$ is an embodiment of the standard Fock space.

4) The φ^4 scalar theory is renormalizable. See, for example: H. Umezawa, Quantum Field Theory, North-Holland Publ. Co., (1956).

5) R. Utiyama and T. Imamura, Prog. Theor. Phys. 9 (1953) 431.